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NESTING OF CYCLE SYSTEMS OF ODD LENGTH

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1. Introduction

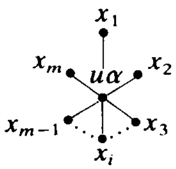
Denote by K_n the complete undirected graph on n vertices. An *m-cycle* of K_n is a collection of m edges $\{x_1, x_2\}$, $\{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$, $\{x_m, x_1\}$ such that the vertices $x_1, x_2, \ldots, x_m$ are *distinct*. In what follows we will denote the m-cycle $\{x_1, x_2\}$, $\{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}$, $\{x_m, x_1\}$ by any cyclic shift of $(x_1, x_2, \ldots, x_m)$. An *m-cycle system* is a pair (K_n, C), where C is a collection of edge disjoint m-cycles which partition K_n. The number n is called the *order* of the m-cycle system (K_n, C) and, of course, the number of m-cycles $|C|$ is $n(n-1)/2m$. A 3-cycle system is, of course, a *Steiner triple system* (everybody's favorite) and a 5-cycle system is a *pentagon system* (well liked by those who know what a pentagon system is).

A *nesting* of the m-cycle system (K_n, C) is a mapping

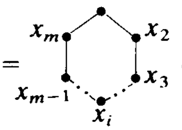
$$\alpha: C \to \{1, 2, 3, \ldots, n\}$$

such that $C(\alpha)$ is an edge disjoint decomposition of K_n where

$C(\alpha) = \Bigg\{$



$\Bigg|\ u =$



$\in C \Bigg\}.$

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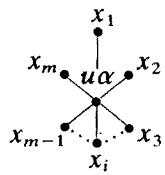
In what follows we will denote the

star 

by $(x_1, x_2, \ldots, x_m; u\alpha)$.

A simple counting argument shows that a necessary condition for an m-cycle system (K_n, C) to be nested is $n \equiv 1 \pmod{2m}$. Whether or not an arbitrary m-cycle system can be nested is undoubtedly an extremely difficult problem. A much more reasonable problem is the following: For a given cycle length m, determine the *spectrum* of m-cycle systems which can be nested (= the set of all $n \equiv 1 \pmod{2m}$ for which there exists an m-cycle system of order n which can be nested). This problem has been completely settled for $m = 3$ [2, 6, 9] (the spectrum for Steiner triple systems which can be nested is precisely the set of all $n \equiv 1 \pmod 6$) *and* with 11 possible exceptions for $m = 5$ [5] (the spectrum for pentagon systems which can be nested is the set of all $n \equiv 1 \pmod{10}$, except possibly 111, 201, 221, 231, 261, 301, 381, 511, 581, 591, and 621).

The purpose of this paper is to prove that for any *odd* cycle length m the spectrum of m-cycle systems which can be nested is the set of all $n \equiv 1 \pmod{2m}$ with *at most* 13 possible exceptions for each m. In addition we remove some of these 13 possible exceptions for small values of m. In particular we remove the possible exceptions for pentagon systems, showing that the spectrum for pentagons systems which can be nested is *precisely* the set of all $n \equiv 1 \pmod{10}$.

Finally, we remark that the nesting of an m-cycle system (K_n, C) is equivalent to an edge disjoint decomposition of $2K_n$ into *wheels*, each with m spokes with the property that for each pair of vertices x and y, one of the edges $\{x, y\}$ occurs on the rim of wheel and one of the edges $\{x, y\}$ is the spoke of a wheel.

In the following, m will always denote a positive ODD integer. Also, when we write $d = i \pmod m$ we assume that $d \in Z_m$.

2. Preliminaries

The main ingredients in our construction of m-cycle systems which can be nested are a *skew Room frame* and an *m-nesting sequence*. We begin with the definition of a skew Room frame.

Let $X = \{1, 2, 3, \ldots, 2s\}$ and let $H = \{h_1, h_2, \ldots, h_t\}$ be a partition of X with the property that each h_i has size 2 or 4. The sets $h \in H$ are called *holes*. Using this jargon, we can say that H is a partition of X into holes of size 2 or 4. Denote by $T(X)$ the set of all 2-element subsets of X and by $T(H)$ the set of all

2-elements subsets belonging to a hole of H. Let F be a $2s \times 2s$ array and fill in (a subset of) the cells of F as follows:

(1) For each hole $h_i \in H$, fill in the cells of $h_i \times h_i$ with

x_1x_2	
	x_1x_2

if $h_i = \{x_1, x_2\}$

x_1x_2	x_3x_4		
	x_1x_2	x_3x_4	
		x_1x_2	x_3x_4
x_3x_4			x_1x_2

if $h_i = \{x_1, x_2, x_3, x_4\}$

(in what follows the cells $h_i \times h_i$, $h_i \in H$, will be called a *square* hole);

(2) distribute the 2-element subsets in $T(X)\setminus T(H)$ among the cells not belonging to a square hole (each 2-element subset used exactly once) so that each row and column of F is a 1-factor of K_{2s}; and

(3) if $\{a, b\} \in T(X)\setminus T(H)$, exactly one of the cells (a, b) and (b, a) of F is occupied.

The resulting array is called a skew Room frame of order $2s$ with holes of size 2 or 4.

Example 2.1

1 2			6 9		8 10		3 5	4 7	
	1 2	6 10		7 9		4 5			3 8
5 10		3 4			2 7		1 9	6 8	
	5 9		3 4	1 8		2 10			6 7
8 9		1 7		5 6			4 10	2 3	
	7 10		2 8		5 6	3 9			1 4
4 6		2 9		3 10		7 8		1 5	
	3 6		1 10		4 9		7 8		2 5
	4 8		5 7		1 3		2 6	9 10	
3 7		5 8		2 4		1 6			9 10

Previous page: Skew Room frame of order 10 with holes of size 2 or 4. (In this example all holes happen to be of size 2.)

We state the following existence theorem for skew Room frames with holes and *delay* the proof until Section 5.

Theorem 2.2. *There exists a skew Room frame in which all holes have size* 2 *for every even order* $n \notin \{2, 4, 6, 8, 12, 44, 46, 48, 52, 54, 56, 60, 68, 76\}$. *There exists a skew Room frame with holes of size* 2 *or* 4 *for every even* $n \notin \{2, 4, 6, 8, 12\}$.

Let $[x]$ denote the greatest integer less than or equal to x and define $D(i, j) = \min\{i - j \pmod m,\ j - i \pmod m\}$. An *m-nesting sequence* is a sequence $(d_0, d_1, d_2, \ldots, d_{[m/2]})$, $i \in Z_m$, such that

(1) $\{D(d_i, d_{i-1}) \mid i = 1, 2, \ldots, [m/2]\} = \{1, 2, \ldots, [m/2]\}$, and

(2) $\{D(d_{[m/2]}, d_i) \mid i = 0, 1, \ldots, [m/2] - 1\} = \{1, 2, \ldots, [m/2]\}$.

Example 2.3

$$\begin{cases} (0, 1) \text{ is a 3-nesting sequence,} \\ (0, 1, 4) \text{ is a 5-nesting sequence,} \\ (0, 1, 6, 2) \text{ is a 7-nesting sequence, and} \\ (0, 1, 8, 2, 7) \text{ is a 9-nesting sequence.} \end{cases}$$

Lemma 2.4. *There exists an m-nesting sequence for every odd* $m \geqslant 3$.

Proof. Define $d_i = (-1)^{i+1}[(i + 1)/2] \pmod m$. Then $(d_0, d_1, d_2, \ldots, d_{[m/2]})$ is an m-nesting sequence. □

We close this section with a construction of an m-cycle system of order $2m + 1$ which, as we shall see in Section 3, is a principal ingredient in the skew Room frame construction.

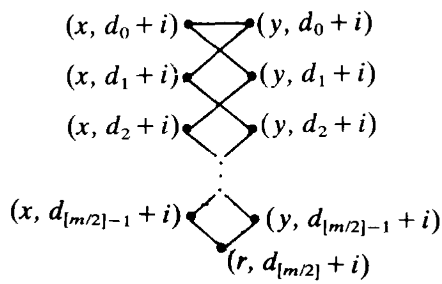
Lemma 2.5. *There exists an m-cycle system of order* $2m + 1$ *which can be nested for every odd* $m \geqslant 3$.

Proof. Let $m = 2n + 1$ and define $c = ((-1)^1 1,\ (-1)^2 \cdot 2, \ldots, (-1)^n \cdot n,\ (-1)^n \cdot (n + 1),\ (-1)^{n+1} \cdot (n + 2), \ldots, (-1)^{2n} \cdot (2n + 1))$, where each coordinate is reduced modulo $2m + 1$. Let $c + i$, $i = 0, 1, 2, \ldots, 2m$, be formed by replacing each coordinate x of c by $x + i \pmod{2m + 1}$. Let K_{2m+1} be based on Z_{2m+1} and define $C = \{c + i \mid i = 0, 1, 2, \ldots, 2m\}$. Then (K_{2m+1}, C) is an m-cycle system of order $2m + 1$ and the mapping α defined by $(c + i)\alpha = i$ is a nesting. □

Example 2.6. For $m = 3$, $c = (6, 5, 3)$, and $C = \{(6 + i, 5 + i, 3 + i) \mid i \in Z_7\}$. For $m = 5$, $c = (10, 2, 3, 7, 5)$ and $C = \{(10 + i, 2 + i, 3 + i, 7 + i, 5 + i) \mid i \in Z_{11}\}$.

3. The skew Room frame construction

We begin with some notation. Let $(d_0, d_1, d_2, \ldots, d_{[m/2]})$ be an m-nesting sequence and $X = \{1, 2, 3, \ldots, 2k\}$. Further let x, y, and r be any 3 distinct elements belonging to X and i any element belonging to Z_m. In what follows we will denote the cycle



by $(x, y, r; d_0 + i, d_1 + i, \ldots, d_{[m+2]} + i)$, where $d_j + i$ is reduced modulo m.

The skew Room frame construction. Let $m \geqslant 3$ be *odd*, $X = \{1, 2, 3, \ldots, 2k\}$, and let K_{2km+1} be based on $\{\infty\} \cup (X \times Z_m)$. Further, let S be a skew Room frame (based on X) with holes H of size 2 or 4 and let $(d_0, d_1, d_2, \ldots, d_{[m/2]})$ be an m-nesting sequence. Now define a collection of m-cycles C of K_{2km+1} as follows:

(1) For each hole $h \in H$, define an m-cycle system (which can be nested) on $\{\infty\} \cup (h \times Z_m)$ and place these cycles in C. (*Important*: If the hole $h \in H$ has size 2, then Lemma 2.5 guarantees the existence of an m-cycle system of order $2m + 1$ which can be nested. It goes without saying that if $h \in H$ has size 4, this construction is used only if it is *known* that an m-cycle system of order $4m + 1$ which can be nested *exists*!); and

(2) for each x and y belonging to *different* holes and *each* $i \in Z_m$, place the m-cycle $(x, y, r; d_0 + i, d_1 + i, \ldots, d_{[m/2]} + i)$ in C, where r is the *row* of S containing the pair $\{x, y\}$.

It is straightforward to see that (K_{2km+1}, C) is an m-cycle system, and so it remains to show that (K_{2km+1}, C) can be nested.

Theorem 3.1. *The m-cycle system (K_{2km+1}, C) constructed using the skew Room frame construction can be nested.*

Proof. For each hole $h \in H$ denote by $h\alpha$ a nesting of the m-cycle system defined on $\{\infty\} \cup (h \times Z_m)$ *and* define a mapping

$$g\alpha = \begin{cases} \text{(1)} \quad g(h\alpha), \text{ if } g \in \{\infty\} \cup (h \times Z_m) \text{ for some } h \in H; \text{ and} \\ \text{(2)} \quad (c, d_{[m/2]} + i), \text{ if } g = (x, y, r; d_0 + i, d_1 + i, \ldots, d_{[m/2]} + i), \\ \qquad \text{where } c \text{ is the } column \text{ of } S \text{ containing } \{x, y\}. \end{cases}$$

Claim: α is a nesting of (K_{2km+1}, C). We must show that the collection of stars $C(\alpha)$ obtained from C is an edge disjoint decomposition of K_{2km+1}. Trivially the m-cycle systems defined on $\{\infty\} \cup (h \times Z_m)$, $h \in H$, are partitioned by stars belonging to $C(\alpha)$ and so it suffices to show that each edge of the form $\{(x, i), (y, j)\}$, x and y in different holes, belongs to some star of $C(\alpha)$. There are two cases to consider: $i = j$ and $i \neq j$.

$i = j$. Let $d_{[m/2]} + t = i = j \pmod m$. Since S is a skew Room frame and x and y belong to different holes, exactly one of the cells (x, y) and (y, x) is occupied. If cell (x, y) is occupied by $\{a, b\}$, then the m-cycle $c = (a, b, x, d_0 + t, d_1 + t, \ldots, d_{[m/2]} + t = i = j) \in C$. Hence the star $((a, d_0 + t), (b, d_0 + t), (a, d_1 + t), (b, d_1 + t), \ldots, (x, d_{[m/2]} + t = i = j); (y, d_{[m/2]} + t = i = j)) \in C(\alpha)$. The same argument is valid if (y, x) is occupied.

$i \neq j$. Let $d = \min\{i - j \pmod m, j - i \pmod m\}$. Then $d \in \{1, 2, 3, \ldots, [m/2]\}$ and so there exists a t such that $D(d_{[m/2]}, d_t) = d$. We assume $d = j - i = d_{[m/2]} - d_t \pmod m$, the other three cases having similar proofs. Then there exists a q such that $j = d_{[m/2]} + q \pmod m$ and $i = d_t + q \pmod m$. Since x and y belong to different holes, column y contains a pair of the form $\{x, z\}$. Denote by (r, y) the cell containing $\{x, z\}$. Then the m-cycle $(x, z, r; d_0 + q, d_1 + q, \ldots, d_{[m/2]} + q) \in C$ and so the star $((x, d_0, +q), (z, d_0 + q), (x, d_1 + q), (z, d_1 + q), \ldots, (x, d_t + q = i), (z, d_t + q = i), \ldots, (r, d_{[m/2]} + q = j); (y, d_{[m/2]} + q = j)) \in C(\alpha)$.

Combining the above two cases shows that the collection of stars $C(\alpha)$ is an edge disjoint decomposition of K_{2km+1} which completes the proof. □

Theorem 3.2. *For any odd $m \geq 3$, the spectrum of m-cycle systems which can be nested is the set of all $n \equiv 1 \pmod{2m}$, with the 13 possible exceptions $n = km + 1$, $k \in \{4, 6, 8, 12, 44, 46, 48, 52, 54, 56, 60, 68, 76\}$.*

Proof. A skew Room frame in which all holes have size 2 exists for every even order $k \notin \{2, 4, 6, 8, 12, 44, 46, 48, 52, 54, 56, 60, 68, 76\}$ (Theorem 2.2). Since there exists an m-cycle system of order $2m + 1$ which can be nested for every odd $m \geq 3$ (Lemma 2.5), the statement of the theorem follows from the skew Room frame construction (Theorem 3.1). □

Corollary 3.3. *If m is odd and there exists an m-cycle system of order $4m + 1$ which can be nested, then the spectrum of m-cycle systems which can be nested is the set of all $n \equiv 1 \pmod{2m}$, with the 3 possible exceptions $6m + 1$, $8m + 1$, and $12m + 1$.*

Proof. In the proof of Theorem 3.2 replace skew Room frames with holes of size 2 with skew Room frames with holes of size 2 or 4. □

4. The spectrum for some small value of m

It should come as no surprise that for a given cycle length m we can improve on the results guaranteed by Theorem 3.2 and Corollary 3.3. We list improvements here for $m \leqslant 15$. There is, of course, nothing special about the number 15. We could just as well use 50 or 100. However, $m \leqslant 15$ is sufficient for illustration.

The principle tool used to improve on the results in Theorem 3.2 and Corollary 3.3 is the finite field construction.

The finite field construction. Let $n = 2km + 1$ be a prime power, x a *primitive* element in $F = \mathrm{GF}(2km + 1)$, and define: $B = \{(x^i, x^{i+2k}, x^{i+4k}, \ldots, x^{i+2(m-1)k}) \mid i = 0, 1, 2, \ldots, k-1\}$. If $b = (a_1, a_2, \ldots, a_m) \in B$ and $y \in F$ denote by $b + y$ the m-cycle $(a_1 + y, a_2 + y, \ldots, a_m + y)$, and set $C = \{b + y \mid b \in B$ and $y \in F\}$. If K_{2km+1} is based on F, then (K_{2km+1}, C) is an m-cycle system and the mapping α given by $(b + y)\alpha = y$ is a nesting.

Finally, we will need the following two m-cycle systems (which can be nested).

(1) Let K_{21} be based on Z_{21} and define $B = \{(1, 6, 19, 18, 7), (4, 16, 13, 9, 11)\}$. Let $C_{21} = \{b + i \mid b \in B$ and $i \in Z_{21}\}$, where $b + i$ is obtained from b by adding $i \pmod{21}$ to each coordinate of b. Then (K_{21}, C_{21}) is a pentagon system and $\alpha: C_{21} \to Z_{21}$ defined by $(b + i)\alpha = i$ is a nesting.

(2) Let K_{45} be based on Z_{45} and define $B = \{(1, 2, 4, 7, 3, 8, 14, 5, 12, 28, 20), (6, 23, 34, 16, 26, 13, 36, 21, 35, 15, 27)\}$. Set $C_{45} = \{b + i \mid b \in B$ and $i \in Z_{45}\}$, where $b + i$ is obtained from b by adding $i \pmod{45}$ to each coordinate of b. Then (K_{45}, C_{45}) is an 11-cycle system and $\alpha: C_{45} \to Z_{45}$ defined by $(b + i)\alpha = i$ is a nesting.

The finite field construction plus (K_{21}, C_{21}) and (K_{45}, C_{45}) guarantees the existence of an m-cycle system of order $4m + 1$ which can be nested for every $m \in \{3, 5, 7, 9, 11, 13, 15\}$. Hence Corollary 3.3 further guarantees for $m \in \{3, 5, 7, 9, 11, 13, 15\}$ that $6m + 1$, $8m + 1$, and $12m + 1$ are the only possible exceptions in the spectrum of m-cycle systems which can be nested. In the following table we have eliminated some of these possible exceptions using the finite field construction.

m	spectrum of m-cycle systems which can be nested
3	all $n \equiv 1 \pmod 6$ Steiner triple systems [9]
5	all $n \equiv 1 \pmod{10}$ pentagon systems [5]
7	all $n \equiv 1 \pmod{14}$ except possibly 57 and 85
9	all $n \equiv 1 \pmod{18}$ except possibly 55
11	all $n \equiv 1 \pmod{22}$ except possibly 133
13	all $n \equiv 1 \pmod{26}$ except possibly 105
15	all $n \equiv 1 \pmod{30}$ except possibly 91

Comments. The spectrum for Steiner triple systems which can be nested was *first* determined by Stinson [9]. The spectrum for pentagon systems was determined with the 11 possible exceptions 111, 210, 221, 231, 261, 301, 381, 511, 581, 591, and 621 by Lindner and Rodger [5]. Denote by $S(m)$ the spectrum of m-cycle systems which can be nested. If $4m+1 \in S(m)$, then $S(m)$ consists of all $n \equiv 1 \pmod{2m}$ with the three possible exceptions $6m+1$, $8m+1$, and $12m+1$ (Corollary 3.3). If $6m+1$, $8m+1$, and $12m+1 \in S(m)$ as well, then $S(m) = \{n \mid n \equiv 1 \pmod{2m}\}$. The important problem of finding a general construction to show that $\{4m+1,\ 6m+1,\ 8m+1,\ 12m+1\} \in S(m)$ remains open. Since it is "surely true" that $\{4m+1,\ 6m+1,\ 8m+1,\ 12m+1\} \in S(m)$ for every odd m, we do not hestitate to make the following *conjecture*: $S(m) = \{n \mid n \equiv 1 \pmod{2m}\}$ *for every odd* m.

5. Proof of Theorem 2.2

We begin with some notation. If S is a skew Room frame with holes H, the *type* of S is defined to be the *multiset* $T(S) = \{|h| \mid h \in H\}$, where $|h|$ is the size of the hole $h \in H$. In what follows we will abbreviate the type $T(S)$ by $1^{t(1)} \cdot 2^{t(2)} \cdot \ldots \cdot k^{t(k)}$, where $t(i)$ denotes the number of holes $h \in H$ of size i, with the proviso that $i^{t(i)}$ occurs in this product if and only if $t(i) \neq 0$. So, for example, a skew Room frame of order 54 with 5 holes of size 2 and 11 holes of size 4 is of type $2^5 \cdot 4^{11}$.

The following result was proved in [10].

Theorem 5.1. *There exists a skew Room frame of type* 2^n *for all* $n \geq 5$, *except possibly for* $n \in \{6, 11, 15, 19, 20, 22, 23, 24, 26, 27, 28, 30, 31, 34, 36, 38, 43, 46, 51, 58, 59, 62, 67\}$.

We prove here the following two results.

Theorem 5.2. *There exists a skew Room frame of type* 2^n *for all* $n \geq 5$, *except possibly for* $n \in \{6, 22, 23, 24, 26, 27, 28, 30, 34, 38\}$.

Theorem 5.3. *For all* $n \geq 5$, $n \neq 6$, *there exists a skew Room frame of order* $2n$, *having holes of size* 2 *and* 4.

In what follows we will *shorten* skew Room frame *to* skew frame. Now, let us recall several constructions from [10]. Let G be an abelian group, written additively, and let H be a subgroup of G. Denote $g = |G|$, $h = |H|$ and suppose that $g - h$ is even. A *frame starter* in $G \backslash H$ is a set of unordered pairs

$S = \{\{s_i, t_i\} \mid 1 \leqslant i \leqslant (g-h)/2\}$ satisfying (1) $\{s_i\} \cup \{t_i\} = G \setminus H$, and (2) $\{\pm(s_i - t_i)\} = G \setminus H$. An *adder* for S is an injection $A: S \to G \setminus H$, such that

$$\{s_i + a_i\} \cup \{t_i + a_i\} = G \setminus H, \quad \text{where } a_i = A(s_i, t_i), \quad 1 \leqslant i \leqslant (g-h)/2.$$

A is *skew* if, in addition, $\{a_i\} \cup \{-a_i\} = G \setminus H$.

Construction 1. Suppose there exists a frame starter S in $G \setminus H$, and a skew adder A for S. Then there is a skew frame of type $h^{g/h}$, where $g = |G|$ and $h = |H|$.

We also use a modified starter-adder construction, which we now describe. As before, let G be an abelian group and let H be a subgroup of G, where $g = |G|$, $h = |H|$, and suppose that $g - h$ is even. A *2k-intransitive starter* in $G \setminus H$ is defined to be a triple (S, R, C), where

$$\begin{cases} S = \{\{s_i, t_i\} \mid 1 \leqslant i \leqslant (g-h-2k)/2\} \cup \{\{u_i\} \mid 1 \leqslant i \leqslant 2k\}, \\ C = \{\{p_i, q_i\} \mid 1 \leqslant i \leqslant k\}, \quad \text{and} \\ R = \{\{p_i', q_i'\} \mid 1 \leqslant i \leqslant k\}, \end{cases}$$

satisfying

$$\begin{cases} (1) \quad \{s_i\} \cup \{t_i\} \cup \{u_i\} \cup \{p_i\} \cup \{q_i\} = G \setminus H, \\ (2) \quad \{\pm(s_i - t_i)\} \cup \{\pm(p_i - q_i)\} \cup \{\pm(p_i' - q_i')\} = G \setminus H, \text{ and} \\ (3) \quad \text{all } p_i - q_i \text{ and } p_i' - q_i' \text{ have even order in } G. \end{cases}$$

An *adder* for (S, R, C) is an injection $A: S \to G \setminus H$, such that $\{s_i + a_i\} \cup \{t_i + a_i\} \cup \{u_i + A(u_i)\} \cup \{p_i', q_i'\} = G \setminus H$, where $a_i = A(s_i, t_i)$, $1 \leqslant i \leqslant (g - h - 2k)/2$. A is *skew* if, further,

(1) $\{a_i\} \cup \{-a_i\} \cup \{A(u_i), -A(u_i)\} = G \setminus H$, and
(2) for each i, $1 \leqslant i \leqslant k$, there exists a $j \geqslant 1$ such that $p_i - q_i$ has order $2^j m_1$ and $p_i' - q_i'$ has order $2^j m_2$, where m_1 and m_2 are odd.

Construction 2. If there is a $2k$-intransitive frame starter and a skew adder in $G \setminus H$, where $g = |G|$ and $h = |H|$, then there is a skew frame of type $h^{g/h}(2k)^1$.

We next describe recursive constructions for skew frames. All required design theoretic terminology can be found in [1].

Construction 3. Let (X, G, B) be a group divisible design (GDD), and let $w: X \to Z^+ \cup \{0\}$ (we say that w is a *weighting*). For every $b \in B$ suppose there is a skew frame of type $\{w(x) \mid x \in b\}$. Then there is a skew frame of type $\{\sum_{x \in g} w(x) \mid g \in G\}$.

Construction 4. Suppose (X, B) is a pairwise balanced design (PBD), and there exists a skew frame of type $2^{|b|}$, for every $b \in B$. Then there is a skew frame of type $2^{|X|}$.

Construction 5. Suppose $m \geq 4$, $m \neq 6$ or 10, and suppose $0 \leq t \leq 3m$. Suppose also that there exist skew frames of types 2^{2m} and 2^t. Then there exists a skew frame of type 2^{8m+t}.

Construction 6. Suppose $s = u(v-1)+1$, and let t be a rational number such that $2t$ and $(v-1)/t$ are both integers. Suppose there exist skew frames of type $(2t)^u$ and 2^v, and suppose that $(v-1)/t \neq 2$ or 6. Then there exists a skew Room frame of type 2^s.

Construction 7. Suppose there is a skew Room frame of type $t_1^{u_1}t_2^{u_2}\cdots t_j^{u_j}$, and suppose also that $t \neq 2$ or 6. Then there exists a skew Room frame of type $(t\cdot t_1)^{u_1}(t\cdot t_2)^{u_2}\cdots(t\cdot t_j)^{u_j}$.

Lemma 5.4. *There is a skew frame of type* 2^{59}.

Proof. This is a special application of Construction 3. We start with a group divisible design (GDD) of type 3^8 having blocks of size 4, in which the blocks can be partitioned into 7 parallel classes (see [4] for a construction of this design). Adding a new infinite point to each of 5 of the parallel classes, we obtain a GDD of group-type $3^8 5^1$ having blocks of size 4 and 5. Give every point weight 4, and apply Construction 3, using input frames of type 4^4 and 4^5 (these are constructed in [7]). A skew Room frame of type $12^8 20^1$ is produced. Now, add on two new rows and columns, and fill in the holes with skew frames of types 2^7 and 2^{11}. A skew frame of type 2^{59} results. □

Lemma 5.5. *There exist skew frames of type* $4^{11}2^1$, 4^{12}, *and* $4^{11}2^5$.

Proof. The constructions are obtained by the methods of "projecting sets" as described in [8]. The frames are all constructed by means of intransitive starters and skew adders, by altering slightly the following starter and skew adder in $G\backslash H$, where $G = Z_{11}\times Z_2\times Z_2$ and $H = \{0\}\times Z_2\times Z_2$. Suppose $S = \{\{(x,0,0),(2x,0,0)\}, \{(x,0,1),(2x,1,0)\}, \{(x,1,0),(2x,1,1)\}, \{(x,1,1),(2x,0,1)\} \mid x = 1,3,4,5,9\}$, and $A((x,i,j),(2x,k,l)) = (x, i+k, j+l)$. Then S and A generate a skew frame of type 4^{11}. Now consider the two pairs (in S) $\{(1,0,1),(2,1,0)\}$ and $\{(3,0,1),(6,1,0)\}$. Suppose we delete these two pairs from S, and adjoin the two singletons $\{(1,0,1)\}$ and $\{(6,1,0)\}$, obtaining S'. Then, define $C = \{(2,1,0),(3,0,1)\}$, and $R = \{(3,0,1),(6,1,0)\}$. This produces a 2-intransitive starter and skew adder (S', R, C), and hence there is a skew frame of type $4^{11}2^1$. Now, repeat the above procedure, starting with S', using the pairs $\{(1,1,0),(2,1,1)\}$ and $\{(3,1,0),(6,1,1)\}$. This gives a 4-intransitive starter and skew adder, producing a skew frame of type $4^{11}4^1 = 4^{12}$. We can do this trick three times more, using pairs $\{(1,1,1),(2,0,1)\}$ and $\{(3,1,1),(6,0,1)\}$; $\{(9,0,1),(7,1,0)\}$ and $\{(5,0,1),(10,1,0)\}$; and $\{(9,1,0),(7,1,1)\}$

and $\{(5,1,0),\ (10,1,1)\}$. Thus we obtain a 10-intransitive starter skew adder, and a skew frame of type $4^{11}10^1$. Filling in the hole of size 10 with a skew frame of type 2^5, we obtain the skew frame of type $4^{11}2^5$. This completes the constructions. □

In a similar fashion, we can prove the following lemma.

Lemma 5.6. *There is a skew frame of type* 4^{14}.

Proof. The procedure is similar to that used in Lemma 5.5. We begin with the following starter and skew adder in $G \backslash H$, where $G = Z_{13} \times Z_2 \times Z_2$ and

Table 1. Constructions for skew frames of type 2^n.

n	Construction	Remark
11	2	Table 3
15	2	Table 3
19	2	Table 3
20	1	Table 3
$31 = 5(7-1) + 1(t = 3/2)$	6	A skew frame of type 3^5 is constructed in [3]
$36 = 5(8-1) + 1(t = 1)$	6	
$43 = 8 \cdot 4 + 11$	5	
$46 = 5(10-1) + 1(t = 1)$	6	
$51 = 5(11-1) + 1(t = 1)$	6	
$58 = 7 \cdot 8 + 1 + 1$	4	There is a PBD on 58 points having blocks of size 7, 8, and 9, constructed by deleting points from a TD(9, 8)
59	Lemma 5.4	
$62 = 7 \cdot 8 + 5 + 1$	4	There is a PBD on 62 points having blocks of size 5, 7, 8, and 9, constructed by deleting points from a TD(9, 8)
$67 = 8 \cdot 7 + 11$	5	

Table 2. Constructions for skew frames with holes of size 2 and 4.

n	Frame	Construction	Remark
44	4^{11}	7 $(t = 4)$	a skew frame of type 1^{11} exists [7]
46	$4^{11}2^1$	Lemma 5.5	
48	4^{12}	Lemma 5.5	
52	4^{13}	7 $(t = 4)$	a skew frame of type 1^{13} exists [7]
54	$4^{11}2^5$	Lemma 5.5	
56	4^{14}	Lemma 5.6	
60	4^{15}	7 $(t = 4)$	a skew frame of type 1^{15} exists [7]
68	4^{17}	7 $(t = 4)$	a skew frame of type 1^{17} exists [7]
76	4^{19}	7 $(t = 4)$	a skew frame of type 1^{19} exists [7]

Table 3. Starter-adder constructions for skew frames of type 2^n.

$n=8$	5	6	12	1	2
	10	12	3	13	15
	14	1	6	4	7
	15	3	7	6	10
	4	9	5	9	14
	7	13	14	5	11
	11	2	1	12	3
$n=11$	1	2	17	18	19
	7	9	5	12	14
	14	17	9	3	6
	11	15	14	5	9
	18	3	13	11	16
	6	12	16	2	8
	8	16	19	7	15
	5		8	13	
	19		2	1	
$C=$	4	13			
$R=$				4	17
$n=15$	1	27	5	6	4
	2	26	19	21	27
	3	25	2	5	27
	4	24	20	24	16
	5	23	15	20	10
	6	22	25	3	19
	13	16	27	12	15
	12	17	6	18	23
	11	18	11	22	1
	10	19	16	26	7
	9	20	21	2	13
	7		4	11	
	15		10	25	
$C=$	8	21			
$R=$				8	9
$n=19$	19	20	31	14	15
	15	22	25	4	11
$n=19$	14	23	19	33	6
	13	24	15	28	3
	12	25	13	25	2
	11	26	9	20	35
	10	27	35	9	26
	1	35	6	7	5
	2	34	14	16	12
	3	33	26	29	23
	4	32	28	32	24
	5	31	3	8	34
	6	30	16	22	10
	7	29	24	31	17
	9		4	13	
	17		2	19	
$C=$	16	21			
$R=$				27	30
$n=20$	14	15	39	13	14
	11	13	33	4	6
	5	8	23	28	31
	33	37	12	5	9
	31	36	27	18	23
	10	16	11	21	27
	27	34	38	25	32
	21	29	18	39	7
	23	32	10	33	2
	12	22	4	16	26
	38	9	21	19	30
	35	7	3	38	10
	17	30	5	22	35
	28	2	6	34	8
	4	19	32	36	11
	25	1	16	1	17
	26	3	26	12	29
	6	24	31	37	15
	39	18	25	24	3

$H=\{0\}\times Z_2\times Z_2$. Suppose $S=\{\{(x,0,0),\ (4x,0,0)\},\ \{(x,0,1),\ (4x,1,0)\},\ \{(x,1,0),\ (4x,1,1)\},\ \{(x,1,1),\ (4x,0,1)\}\mid x=1,2,3,5,6,9\}$,

$$\begin{cases} A((x,i,j),\ (4x,k,l))=(3x,\ i+k,\ j+l), & \text{if } x=1, 3, \text{ or } 9, \text{ and} \\ A((x,i,j),\ (4x,k,l))=(10x,\ i+k,\ j+l), & \text{if } x=2, 6, \text{ or } 5. \end{cases}$$

Then S and A generate a skew frame of type 4^{13}. Now, consider the pairs (in S) $\{(5,0,1),\ (7,1,0)\}$ and $\{(1,0,1),\ (4,1,0)\}$. Delete these two pairs from S, and adjoin the two singletons $\{(5,0,1\}$ and $\{(4,1,0)\}$, obtaining S'. Then, define $C=\{(1,1,0),\ (7,0,1)\}$, and $R=\{(4,0,1),\ (5,1,0)\}$. Then, repeat this process, using instead $\{(5,1,0),\ (7,1,1)\}$ and $\{(1,1,0),\ (4,1,1)\}$. This gives a 4-intransitive starter and skew adder, giving rise to a skew frame of type 4^{14}. □

We present in Table 1 a list of skew frames of type 2^n obtained using the above constructions. As an immediate consequence of Theorem 5.1 and Table 1, we obtain Theorem 5.2. As well, we present in Table 2 a list of skew frames with holes of size 2 and 4. As an immediate consequence of Theorem 5.2 and Table 2, we obtain Theorem 5.3. Theorem 2.2 is, of course, the combination of Theorems 5.2 and 5.3.

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